Programming theStiffness Method for Pin-Jointed Frames

$ORIGIN := 1$

The following Mathcad sheet details the programming steps for any pin jointed frame. There are variations in the methodology which tend to relate to the programming software used, but Mathcad illustrates the process in a logical manner. Note that any user input is shown here in greyed boxes.

1. Data Required for 2D Pin-Jointed Frame:

Basic data for the frame needs to be input here. Firstly The number of nodes and number of members needs to be input. Number ranges are then defined for constructing stiffness matrices etc. Node locations are required, and a connectivity pattern defined also.

Enter the number of Nodes $nn := 4$ node range $i := 1, 2 .. nn$

Enter the number of Members $nm := 5$ member range $j := 1, 2 .. nm$

DOF range (2 x nn) $k := 1, 2 .. 2 \cdot nn$

Enter the node locations

$$x := \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \cdot m \qquad y := \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} \cdot m$$

1.1 Connectivity

The matrix 'CON' defines the connectivity of the elements to the nodes. The first column refers to the member numbers. The second and third columns correspond to the node numbers to which the members in the first column are connected to. Remember, insert the lower node number first.

$$CON := \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \\ 4 & 2 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

1.2 Section Properties - Young's Moduli and Cross Sectional Areas

Enter the Young's modules 'E' and cross sectional area 'A' values of each Member (note that the matrix row location refers to the associated member number)

$$Area := \begin{pmatrix} 500 \\ 500 \\ 750 \\ 750 \\ 500 \end{pmatrix} \cdot mm^2 \qquad E := \begin{pmatrix} 200 \\ 200 \\ 200 \\ 200 \\ 200 \end{pmatrix} \cdot kN \cdot mm^{-2}$$

2. Stiffness Calculations

The stiffness calculations can now proceed logically as per the hand calculations, however all nodes and degrees of freedom are considered to ensure the procedure is completely general.

2.1 Calculation of member lengths

The length of each member can be calculated simply from Pythagorus, i.e. use the 'x' amd 'y' locations from the CON matrix.

Lengths
$$L_j := \sqrt{\left[x_{\left(CON_{j,3}\right)} - x_{\left(CON_{j,2}\right)}\right]^2 + \left[y_{\left(CON_{j,3}\right)} - y_{\left(CON_{j,2}\right)}\right]^2} \qquad L = \begin{pmatrix} 3 \\ 3 \\ 4.243 \\ 3 \\ 4.243 \end{pmatrix} m$$

2.2 Member Stiffness Matrices

The 2x2 member stiffness matricesare easilt assembled for each member 'j'

Stiffness matrices
$$K_j := \frac{E_j \cdot Area_j}{L_j} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

2.3 Member angles measured from X-axis

Next we need to calculate the angle from the x-axis which each member subtends. this is measured positive in an anti-clockwise sense. These angles will be required to calculate the transformation matrices

Angles with the x-axis
$$\theta_j := \text{if}\left[x_{\left(CON_{j,3}\right)} - x_{\left(CON_{j,2}\right)} > 0, \text{asin}\left[\frac{y_{\left(CON_{j,3}\right)} - y_{\left(CON_{j,2}\right)}}{L_j}\right], 180 \cdot deg - \text{asin}\left[\frac{y_{\left(CON_{j,3}\right)} - y_{\left(CON_{j,2}\right)}}{L_j}\right]\right]$$

Check Member angles:
$$\theta = \begin{pmatrix} 180 \\ 270 \\ 225 \\ 180 \\ 135 \end{pmatrix} \cdot deg$$

2.4 Transformation matrices

The following code creates 'nm' transformation matrices (2 x 2nn), inserting the cos/sin functions at the appropriate locations which link local to global degrees of freedom:

$$t_j := \left| \begin{array}{l} \text{for } z \in 1..2\cdot nn \\ \quad \left| \begin{array}{l} temp_{1,z} \leftarrow \left| \begin{array}{ll} \cos(\theta_j) & \text{if } z = 2\cdot CON_{j,2} - 1 \\ \sin(\theta_j) & \text{if } z = 2\cdot CON_{j,2} \\ 0 & \text{otherwise} \end{array} \right. \\ temp_{2,z} \leftarrow \left| \begin{array}{ll} \cos(\theta_j) & \text{if } z = 2\cdot CON_{j,3} - 1 \\ \sin(\theta_j) & \text{if } z = 2\cdot CON_{j,3} \\ 0 & \text{otherwise} \end{array} \right. \end{array} \right. \\ temp \end{array} \right.$$

Check the matrices :

$$t_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$t_3 = \begin{pmatrix} -0.707 & -0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & -0.707 \end{pmatrix} \quad t_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$t_5 = \begin{pmatrix} 0 & 0 & -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.707 & 0.707 & 0 & 0 \end{pmatrix}$$

Comparing with the hand calc, you can see that the boundary DOF's are also included. Boundaries conditions are dealt with later on.

2.5 Assemble Global stiffness matrix

Use assembly equation to create global stiffness matrix $$KG := \sum_{j=1}^{nm} \left(t_j^{\,T} \cdot K_j \cdot t_j \right)$$

$$KG = \begin{pmatrix} 51.011 & 17.678 & 0 & -0 & -33.333 & 0 & -17.678 & -17.678 \\ 17.678 & 51.011 & -0 & -33.333 & 0 & 0 & -17.678 & -17.678 \\ 0 & -0 & 61.785 & -11.785 & -11.785 & 11.785 & -50 & 0 \\ -0 & -33.333 & -11.785 & 45.118 & 11.785 & -11.785 & 0 & 0 \\ -33.333 & 0 & -11.785 & 11.785 & 45.118 & -11.785 & 0 & 0 \\ 0 & 0 & 11.785 & -11.785 & -11.785 & 11.785 & 0 & 0 \\ -17.678 & -17.678 & -50 & 0 & 0 & 0 & 67.678 & 17.678 \\ -17.678 & -17.678 & 0 & 0 & 0 & 0 & 17.678 & 17.678 \end{pmatrix} \cdot kN\cdot mm^{-1}$$

This matrix is singular at present, boundary conditions will be required to remove the singularity and prevent rigid body motion

2.6 Applied forces

Assemble the Global Force Matrix (force annotation same as for DOF notation)

$$P := \begin{pmatrix} 0 \\ -10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot kN$$

2.7 Boundary conditions

The following matrix defines the fixity for each degree of freedom. For the variable 'BC' insert a '0' if unrestrained, insert a '1' if restrained.

$$BC := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Now remove singularity of global stiffness matrix by adding in a matrix with large stiffness values along the leading diagonal at locations where the degree of freedom is restrained. Add a stiffness of say 10^{50} kN/mm:

Matrix to be added: $KGBC_{k,k} := \text{if}\left(BC_k = 0, 0, 10^{50} \cdot kN \cdot mm^{-1}\right)$

$$KGBC = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \times 10^{50} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \times 10^{50} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \times 10^{50} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \times 10^{50} \end{pmatrix} \cdot kN \cdot mm^{-1}$$

Modded global stiffness matrix to include BC's : $KG_{mod} := KGBC + KG$

3 Solution

3.1 Calculation of Displacements

Normally this is the part of the calculation which requires the solution of simultaneous equations using Gauss Elimination, and requires the lions share of the programming. Luckily in Mathcad we can calculate the inverse of the stiffness matrix directly:

$$d := \left(KG_{mod}\right)^{-1} \cdot P$$

$$d = \begin{pmatrix} 0.183 \\ -0.527 \\ -0.078 \\ -0.41 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot mm$$

Compare with the results from the notes

3.2 Calculation of Member Forces

$$F_j := K_j \cdot t_j \cdot d$$

$$\left(F_j\right)_2 =$$

6.09
-3.91
-8.613
-3.91
5.529

$\cdot kN$

Note: +ve tension
-ve compression

4. Conclusion

The procedure presented is applicable for any pin jointed frame which is joint loaded. It can be expanded to incorporate members with more degrees of freedom, i.e. rotational and shear DOF's, and also into 3D structures.