

Study 8

Live kinematic schemes

Or

Arm of Hella¹

The Study explains how it is possible to create animated clips based on mathematical models of various mechanisms and how to use Mathcad website for educational work. The issues of optimizing the operation of mechanisms are considered. A new type of visual art based on mathematical models and animations of kinematic mechanisms is discussed.

Mathematics: equation of the circle, equation of the straight line, system of linear algebraic equations.

Physics: steam engine, car wipers, lambda-mechanism, plantigrade machine, shock absorber.

IT: tools of the computer analytical transformations, variable range, animation, approximate solution of a system of equations, combination of symbolic and numerical solutions, Mathcad user forum.

Art: Mikhail Bulgakov "Master and Margarita", Fedor Dostoevsky "The Gambler", new form of fine art

¹ Hella is a fictional character from the novel "Master and Margarita" by the Russian writer Mikhail Bulgakov. The novel was written between 1928 and 1940, but unpublished in book form until 1967. Hella is a beautiful, redheaded succubus. She serves as maid to Woland and his retinue. Described as being "perfect, were it not for a purple scar on her neck" — the scar suggesting that she is also a vampire. Reference to her arm will be clear later in this Study.

Study website: <https://community.ptc.com/t5/PTC-Mathcad/Chapter-8/m-p/607677>

In 1705, the English ironmonger Thomas Newcomen¹ built a steam engine, that was used for pumping out water from mines². It was a so-called atmospheric-type steam engine. Saturated water vapor was supplied from the boiler to the vertical cylinder. When the piston reached the upper limit point, the steam supply valve was closed and another valve was opened, supplying cold water to the cylinder. The steam condensed and the piston reversed. Then the water from the piston was poured out, fresh steam was fed from the boiler and everything was repeated.

The Study website contains the author's animation of this steam engine, whose two frames are shown in Figure 8.1.

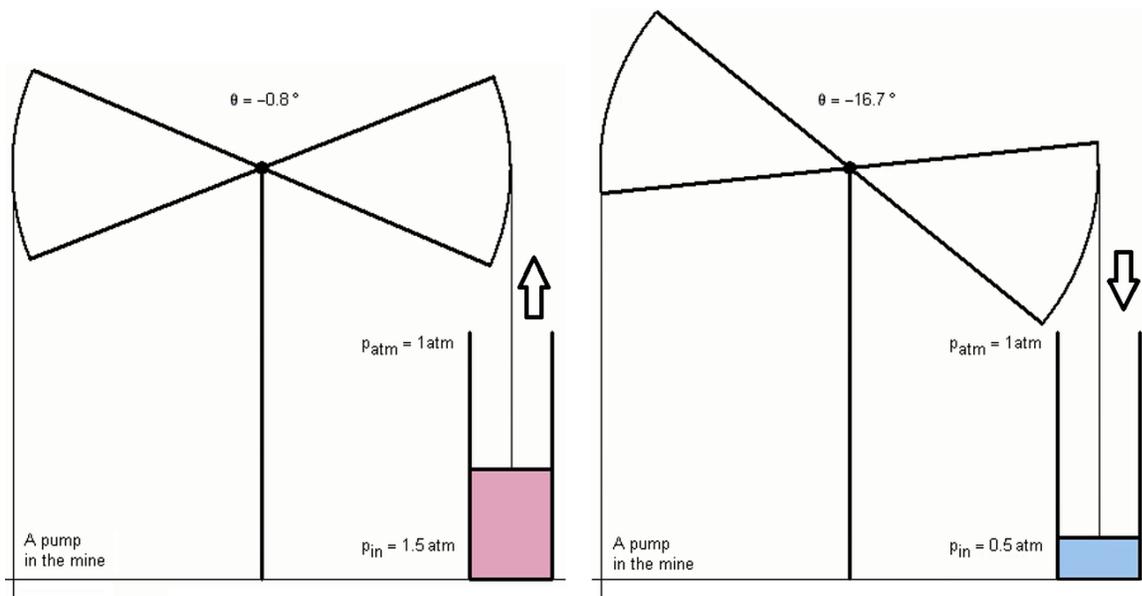


Fig. 8.1. Two frames of animation of the atmospheric steam engine

There is a legend connected with this machine. In 1712 the boy Humphrey Potter was

¹ As Wikipedia mentions, the inventor Thomas Newcome was an ironmonger by trade and a Baptist lay preacher by calling.

² Water will also accumulate in the rectilinear tunnel of the gravity train, which is described in study 9, and there will be the need to pump the water out.

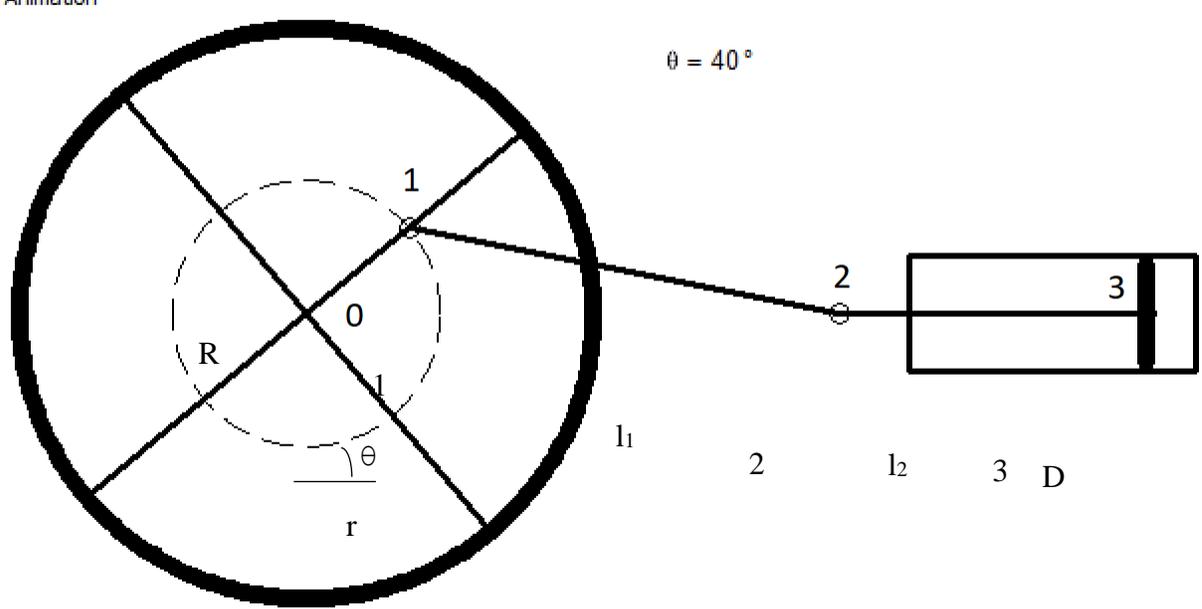
appointed as operator of this type of steam pump. To carry out the working cycle, he had to open the steam supply valve when the piston was in its lower position, while he had to open the water supply valve when the piston was fully raised. However this clever boy figured out how to get rid of this monotonous work¹. With the help of chords and catches the boy connected the valves handles to the beam of the steam engine. It turned out a kind of wedge mechanism. Now the valve was controlled by the piston itself, and the boy could play with his comrades in the yard. Thus, the *slide valve* was invented — an automatic steam distribution device for the steam engine².

A steam engine "in iron" can now be seen only in museums. On the Internet, there are many animations of the steam engine itself or installed on a locomotive or on a steamboat. With the help of the mathematical program Mathcad it is quite possible to create the animation of a steam engine. This feature is discussed in this Study.

¹ In the book of Mark Twain, *Tom Sawyer*(Chapter Two) there is a piece of story in which Tom Sawyer was able to get rid of the monotonous whitewashing fence job: http://www.pbs.org/marktwain/learnmore/writings_tom.html Although Tom made use of political (soft-power) skills instead of the technical acumen shown by Humphrey.

² A "zolotnik" (orig. Russian, a slide valve meaning literally "a golden one") is also an old Russian unit of mass — see Fig. 6.7 in Study 6.

Solution
 Point 0 $y_0 := 0m$ $x_0 := 0m$
 Point 1 $y_1 := r \sin(\theta)$ $x_1 := r \cos(\theta)$ Point 2 $y_2 := 0m$
 $l_1 := l_1$ $y_1 := y_1$ $x_1 := x_1$ $y_2 := y_2$
 $l_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ solve, $x_2 \rightarrow \begin{pmatrix} x_1 + \sqrt{l_1 + y_1 - y_2} \sqrt{l_1 - y_1 + y_2} \\ x_1 - \sqrt{l_1 + y_1 - y_2} \sqrt{l_1 - y_1 + y_2} \end{pmatrix}$
 $x_2 := x_1 + \sqrt{l_1 + y_1 - y_2} \sqrt{l_1 - y_1 + y_2}$
 Point 3: $y_3 := 0m$ $x_3 := x_2 + l_2$
 Steam cylinder $\Delta := 50mm$ $yc := \begin{pmatrix} \frac{D}{2} + \Delta \\ -\frac{D}{2} - \Delta \\ -\frac{D}{2} - \Delta \\ \frac{D}{2} + \Delta \\ \frac{D}{2} + \Delta \end{pmatrix}$
 $\theta_1 := 0^\circ, 1^\circ \dots 360^\circ$
 $xc := \begin{pmatrix} r + l_1 + l_2 + \Delta \\ r + l_1 + l_2 + \Delta \\ -r + l_1 + l_2 - \Delta \\ -r + l_1 + l_2 - \Delta \\ r + l_1 + l_2 + \Delta \end{pmatrix}$

Solution
 Animation


Animation

Fig. 8.2. Mathcad-document, drawing of a steam engine

Figure 8.2 shows a Mathcad document, a drawing and then an animated horizontal steam engine,

or rather its central part consisting (see left to right) of the flywheel of radius R , the connecting rod (rod 1-2) of length l_1 , the shaft (rod 2-3) of length l_2 , the cylinder and the piston.

At the beginning of the calculation, the input data are set with the help of the assignment operator ($:=$): the radius R of the flywheel, the radius r of the circle, along which the left end of the connecting rod (point 1) rotates, the length of the connecting rod l_1 , the length of the rod l_2 and the diameter of the piston D . After this, the angle of rotation θ of the flywheel. In Figure 8.2 this value is specified by two operators $\theta = \text{FRAME}^\circ$ and $\theta = 40^\circ$. For debugging the program the second operator is used that allows the user to manually change the angle of rotation of the flywheel in the range from 0° to 360° and to see how the relative positions of the parts of the steam engine change. When creating the animation (as will be discussed in detail below), the operator $\theta = 40^\circ$ is disabled or erased altogether, so that the first operator $\theta = \text{FRAME}^\circ$ works. The operator is disconnected using a special switch that appears after clicking the right mouse button when the cursor is on the given operator. When the operator is switched off from calculation there is a label — a black square on the right side. The system variable FRAME is zero by default, therefore, the value of the angle θ is zero. However when creating the animation, the Mathcad package itself changes the value of the FRAME variable (see below), setting the flywheel to a different angle: 0° , 1° , 2° , etc. up to 360° .

The kinematic scheme of our steam engine has four joints, which we numbered as 0, 1, 2 and 3. The first joint is fixed. The vertical coordinates of the joints 2 and 3 will remain unchanged. At the zero joint, we place the origin of the coordinates: $y_0 = 0\text{m}$ $x_0 = 0\text{m}$. (The variables y_0 and x_0 are not zero values of the y and x vectors, but variables with a lower index named y_0 and x_0 , where the zeros (indices) are only lowered slightly down. In Mathcad, lower indexes of variables are entered using a point in the variable name $y.0$ and $x.0$. When the user moves the cursor

away from this variable name, the point disappears, and the index moves down.) The coordinate system is Cartesian. However some other mechanical schemes (the pendulum swing model are better "placed" in the polar coordinate system, which is also available in Mathcad. The correct choice of the origin and of the coordinate system can greatly simplify the solution of a problem. It is also very important to determine the joints of a kinematic scheme and their numbering. Our problem is planar, but in Mathcad there is also the three-dimensional graphics, which allows the realization of 3D kinematic schemes with a third coordinate z .

To obtain the scheme of the steam engine, it is necessary to calculate the coordinates of the points 1, 2 and 3, depending on the value of the angle θ . This operation is done in the Mathcad document area (Figure 8.2) denominated **Solution**. The operators specifying the coordinates of the points are enclosed in an *area* that could be hidden in order to see the scheme of the steam engine right after the operators for specifying the angle θ of rotation of the flywheel. This method simplifies the process of debugging the program: it is not necessary to change the value of θ , and then scroll the display screen vertically to see the picture. An area collapses with the command **Collapse** from the list that appears when the right mouse button is pressed. While an area is insert into the calculation with the command **Insert** from the same menu. The area can also be protected from editing (with or without a password).

A more or less complicated calculation is required for determining the abscissa of point 2 — the values of the variable x_2 : analytically (symbolically) the equation of the length of the connecting rod is solved. This nonlinear equation, following from the Pythagorean theorem, has two solutions (**roots**) — the connecting rod with the specified length l_1 and the coordinates of the fixing point on the flywheel (x_1 and y_1) can be either on the left or on the right of the flywheel. The operator of symbolic mathematics **solve**, returns two x_2 solutions in form of a vector. We

take the first element of this vector as the solution of our problem — the coordinate x_2 . Operators $l_1:=l_1$, $y_1:=y_1$, $x_1:=x_1$ and $y_2:=y_2$, written below before the `solve` operator, "suppress" the numerical values of the variables l_1 , y_1 , x_1 and y_2 so that the `solve` operator returns expressions for x_2 , rather than the numerical values of these expressions. Symbolic mathematics operators (in Figure 8.2 they are kept on a gray background) can be removed from the calculation after their use. However, we left them in order to explain this step. In truth, the equation of the rod position can also be solved mentally without involving computer mathematics of analytical (symbolic) transformations. Nonetheless we used these tools to show how symbolic mathematics helps to solve problems in the Mathcad package environment.

The steam cylinder is a rectangle that we draw through its four vertices. Vectors x_c and y_c store the coordinates of these vertices. The variable Δ is the value of the "clearance" of the cylinder: the cylinder length is greater than the extreme points of the piston stroke by Δ , and the diameter of the cylinder is larger than the piston diameter by 2Δ .

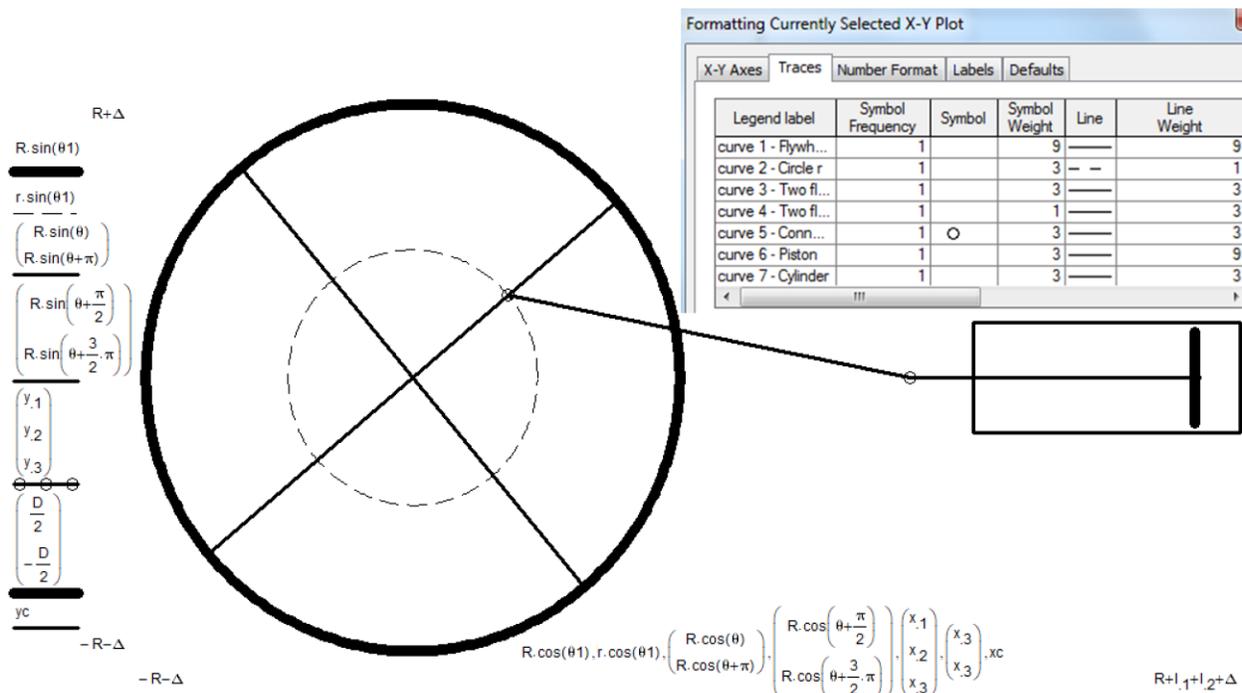


Fig. 8.3. Formatting the chart — steam engine scheme

Figure 8.3 shows the process of inserting, filling, and formatting the Cartesian graph, to draw the steam engine. The graph has seven curves (and all there may be up to sixteen of them), whose description (seven pairs of expressions) is represented on the chart axes and listed in the chart formatting window:

Curve 1. The circumference of the flywheel of the steam engine is drawn as follows. The variable $\theta 1$ is set, storing not a scalar value but discrete values of the angle from 0° to 360° in steps of 1° (see the operator at the bottom of the solution in Figure 8.2). A so-called parametric curve is constructed on the graph (Figure 8.3) with the expressions $R \cdot \sin(\theta 1)$ on the Y axis and $R \cdot \cos(\theta 1)$ on the X axis, which allows to draw a circle (in the figure this is still an ellipse) of radius R . If the variable $\theta 1$ stores discrete data not along the entire circumference, but only on a part ($\theta 1: = 40^\circ, 41^\circ \dots 90^\circ$, for example), then not the whole circle will be drawn, but only an arc. Such dotted arcs can be seen in Figure 8.6 below.

Curve 2. The circumference of the motion of the joint (point 1) of the connecting rod on the flywheel is constructed in the same way as the circle of the flywheel, except that this circle is smaller and it is drawn with a thin dashed line instead of a continuous thick line. The centers of our two circles with radii R and r are at the origin. However if we add constants to the expressions $R \cdot \sin(\theta 1)$ and $R \cdot \cos(\theta 1)$, then the circle will be drawn with a displacement relative to the origin. If we do not add constants to the expressions $R \cdot \sin(\theta 1)$ and $R \cdot \cos(\theta 1)$, but quantities that depend on the variable θ (animation variable), then the circle will move. For example, it is possible to animate a rolling wheel.

Curves 3-7. The spokes of the flywheel of the steam engine are drawn in order to "fix" the connecting rod on the flywheel and show the rotation of the flywheel during the animation. We limited ourselves to four spokes (two crossed diameters), but more spokes can be drawn. In

Study 7 the animated wheel of the bicycle has 8 spokes. The spokes, like other straight segments, as well as broken lines consisting of straight lines, are drawn by assigning a pair of vectors that store the coordinates of the points: two points for the spokes and the piston, three points for the connecting rod and the piston rod (points 1, 2 and 3) and five points for the steam cylinder. We placed the setting of the pair of vectors for the cylinder, in the **Solution** area in order not to clutter the graph area.

The size of the graph should be defined so that the flywheel looks round (see Figure 8.2, in Figure 8.3, we repeat, the flywheel looks oval due to the fact that the proportions of the graph are somewhat distorted when it is formatted). Changing the size of the chart is carried out on by stretching its frame. In addition, when creating an animation, it is necessary to fix the ranges of the graph along its axes. By default, the Mathcad package itself sets the ranges, according to the maximum and minimum values for the abscissa and ordinate axes, which can vary. However in the process of creating an animation, these ranges must be fixed: at the end of the graph we specified $-R-\Delta$ and $R+\Delta$ (X axis) and $-R-\Delta$ and $R+l_1+l_2+\Delta$ (Y axis), where Δ is a certain gap not only for the cylinder of the steam engine (see above) but also for the graph. If the ranges of the chart are not fixed, then the animation will be "curved": the circle will deform into an ellipse, etc.

Before the era of cinema and television, there was a popular toy consisting of a notebook with a man drawn in several pages in different poses. If the notebook was simply scrolled, then the movement of the little man (animation) would not be seen. But if the pages of the notebook were bent and quickly released at once, then there would be an animation — the drawn little man would start moving¹. The same situation could be observed with our calculation of the geometric

¹About this toy, we will tell more in Study 13

parameters of the steam engine. If somebody manually changes the value of the variable θ and collapses the **Solution** area, he may see how the position of the piston, rod, connecting rod and flywheel changes (this is done, we repeat, when debugging the program), but the animation does not work. Animation occurs when the frame rate is at least ten frames per second. For "flipping through" our "notebook" in Mathcad with images of the steam engine in different phases of the motion *special tools for creating animations* were called — see Figure 8.4.

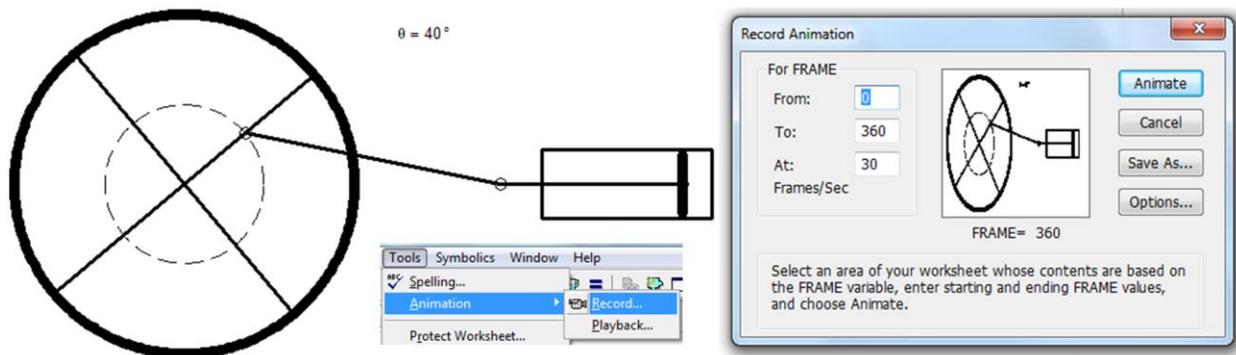


Fig. 8.4. Creating the Steam Engine Animation

To create the animation, first, it is necessary to transfer the control of the independent animation variable (in this case, the variable θ — the angle of rotation of the flywheel) to the system variable **FRAME** (animation frame counter). This is done, we repeat, by disabling the operator $\theta := 40^\circ$. Secondly, it is necessary to call up the animation creation dialog box (Figure 8.4) and to set the necessary animation parameters in this window, and thirdly, to drag the mouse to select the animation area. In Figure 8.4 this area is marked with a dotted line. When an animation is created, it is displayed in the **Record** animation window. Now, after clicking the **Animate** button (see Figure 8.4), the Mathcad package itself changes the value of the angle of rotation of the flywheel of the steam engine θ by the system variable **FRAME** and stores the animation frames. When the animation is complete, a window with the animation appears — see Figure

8.5. This animation could be run in Mathcad, and it is possible to save it as an avi-file on the disk and to run the file without Mathcad in the environment of any player of Windows animation.

That's all! The steam engine was created, and we felt ourselves as some kind of "creator", who can breathe life into immovable virtual objects.

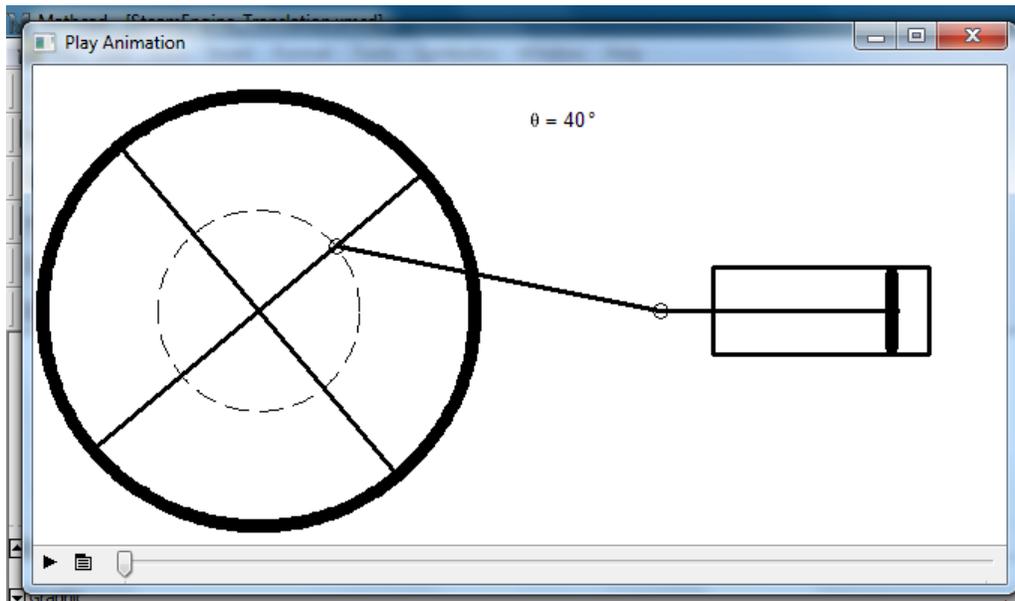


Fig. 8.5. Animation of the steam engine

The reader can complicate the steam engine — add to it the already mentioned legendary slide device. But if this seems too complicated, it is possible to first practice on another kinematic problem — the problem of the motion of the wipers of a car. While the steam engine could be seen only in a museum or on the Internet, we bump into car wipers almost every day.

Car wipers are also surrounded by legends¹. It is told how after a concert a conductor came back home in the rain. The driver often had to stop the car to wipe the windshield. And the conductor at that time still "lived" his concert, beat in his mind musical beat and saw before him a

¹ An interesting movie about windshield wipers: [https://en.wikipedia.org/wiki/Flash_of_Genius_\(film\)](https://en.wikipedia.org/wiki/Flash_of_Genius_(film)), where the college engineering professor Robert Kearns invented a wiper blade mechanism modeled on the human eye, which blinks every few seconds rather than continuously i.e the intermittent windshield wiper systems used on most automobiles from 1969 to the present"

"metronome" — a hand that wipes rain drops from the glass. The next day, it was installed on the hood of the car not an imaginary but a real metronome with a brush attached to it. In this way, they say, the car windshield wipers were invented.

In the era of the steam engines, the problem was to convert the translational motion into rotational motion. When solving the problem of the steam engine (Figures 8.2-8.5), we slightly cheated. By changing the angle of rotation of the flywheel, it was not the translational movement of the piston that was converted into the rotational movement of the flywheel, but the rotational movement of the flywheel was converted into the translational movement of the piston. We actually implemented not a steam engine, but a piston compressor driven by an electric motor or an internal combustion engine. Nevertheless, we have got the animation, and to find out "who is driving whom" is not so important.

At present, in connection with the widespread distribution of electric motors with a rotor and a stator, the problem of converting rotational motion to the translational one has become more urgent. Car wipers are a typical example of such a conversion.

Point 0 $x_0 := 0\text{mm}$ $y_0 := 0\text{mm}$ Point 1 $x_1 := r \cos(\theta) = 60\text{mm}$ $y_1 := r \sin(\theta) = 0\text{mm}$
 Point 5 $x_5 := b = 150\text{mm}$ $y_5 := h = 240\text{mm}$

Point 2: We solve the system of two algebraic equations

Given

Initial values $x_2 := b + 30\text{mm}$ $y_2 := h$

$$L^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \quad r_1^2 = (x_5 - x_2)^2 + (y_5 - y_2)^2$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \text{Find}(x_2, y_2) = \begin{pmatrix} 225.44 \\ 213.377 \end{pmatrix} \text{mm}$$

Point 3 $x_3 := x_2 - a - b$ $y_3 := y_2$

Point 4 $x_4 := x_5 - a - b$ $y_4 := y_5$

Point 6 on a straight line with points 3 and 4: $y = A + Bx$

$$\begin{pmatrix} A \\ B \end{pmatrix} := \text{Isolve}\left[\begin{pmatrix} 1 & x_4 \\ 1 & x_3 \end{pmatrix}, \begin{pmatrix} y_4 \\ y_3 \end{pmatrix}\right]$$

Given

Initial values $x_6 := x_4$ $y_6 := y_4 + L_1$

$$L_1^2 = (x_4 - x_6)^2 + (y_4 - y_6)^2 \quad y_6 = A + Bx_6$$

$$\begin{pmatrix} x_6 \\ y_6 \end{pmatrix} := \text{Find}(x_6, y_6) = \begin{pmatrix} -1254.4 \\ 506.233 \end{pmatrix} \text{mm}$$

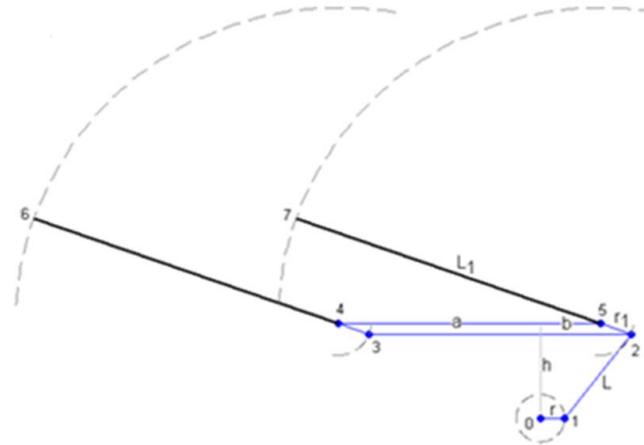


Fig. 8.6. The problem of car windshield wipers

Figure 8.6 shows the calculation using Mathcad 15 of the joints of the kinematic scheme of the car wipers (a parallelogram, in which one of the vertices is moved by the crank), depending on the angle of rotation of the electric motor. The core of the calculation is the determination of the coordinates of joint 2. Here it will be necessary to solve not an equation, as it was in the case of the steam engine (Figure 8.2), but a *system* of two nonlinear algebraic equations. And we will not solve this system analytically (symbolically, since we can try to solve it analytically, however the solution will be very cumbersome), but numerically by the **Given-Find** block built into Mathcad (see Figure 8.6). The numerical solution of the system of equations requires a first approximation to the solution, which is provided by the operators $x_2 := b$ and $y_2 := h$. It is very important to set

the correct first approximation so that the Given-Find block produces the "correct" solution in the entire range of the angle θ change from 0° to 360° . After all, again, the wiper "rod" (line 1-2) with an erroneous first approximation may be on the "erroneous" side of point 5. In this case, the motion of the car wipers may be accompanied by "convulsions" — the connecting rod will appear to the right, then to the left of the attachment point. In principle, the first approximation to the solution should not be a constant, but a variable that depends on the animation variable θ .

Point 6 lies on a straight line with points 3 and 4. Here we first have to solve the system of two linear algebraic equations using the `lsolve` function

$$y_3 = A + B \cdot x_3$$

$$y_4 = A + B \cdot x_4$$

and to determine the values of the coefficients A and B ¹, and then to solve another system of two equations, one of them is nonlinear, and to find the coordinates of point 6. When the coordinates of all points are defined, it is easy to create an animation of the car wipers motion according to the technique described above for the steam engine — see Figures 8.3 and 8.4.

At the end of the 19th century, at the World Exhibition in Paris, the plantigrade machine of the Russian mathematician Pafnuty Tchebyshev received a gold medal.² In this machine, the rotational motion turned into a motion similar to the one of the foot when walking on an even

¹ We solved this problem in Study 2 (see Figure 2.4) in the Mathcad Prime environment, where the dimensionality mechanism works completely and arrays can store different-dimensional quantities, for instance a dimensionless value and a length. In the problem of the wipers of Fig. 8.6 it is necessary to disconnect the tools of dimensional values. This nuance is examined in an example about solving differential equations in Study 9.

²In Mathcad, by the way, there is a built-in function bearing the name of this famous mathematician — Tcheb. From Wiki (https://en.wikipedia.org/wiki/Pafnuty_Chebyshev) "Tchebyshev is considered to be a founding father of Russian mathematics. Among his well-known students were the mathematicians Dmitry Grave, Aleksandr Korkin, Aleksandr Lyapunov, and Andrei Markov. According to the Mathematics Genealogy Project, Tchebyshev has 12,422 mathematical "descendants" as of July 2018".

surface.

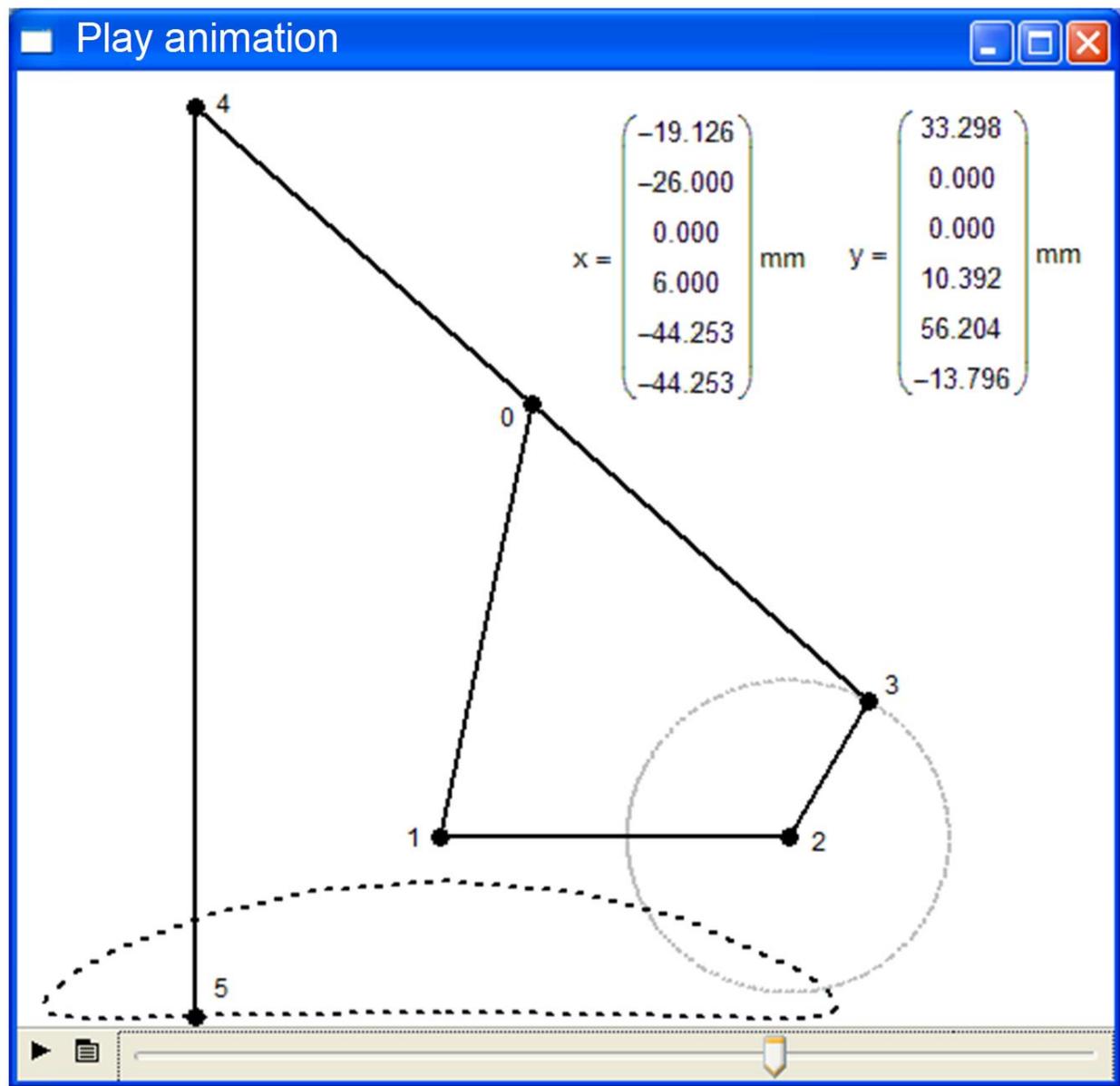


Fig. 8.7. The Tchebyshev plantigrade machine

Figure 8.7 shows the kinematic diagram of a "leg" of the Tchebyshev's plantigrade machine, as well as the trajectory of point 5 and the printout of the vectors x and y — coordinates of six mechanisms points (joints), two of them (1 and 2) are fixed with respect to the origin located at the second point. The numbering of points, again, starts from zero and goes in such a way that the "foot" of Tchebyshev's plantigrade machine can be drawn without taking the pencil from the

paper and not drawing a line twice. This allows to represent a closed curve on the graph using a pair of vectors x and y . The segment of the straight line 2-3 performs a circular motion around point 2. The coordinates of the point 0 can be found by solving a system of two algebraic equations, as we did in the problem of the car wipers (Figure 8.6). We invite the reader to write the Mathcad-document for creating the animation of the "Tchebyshev's leg" by the methods described above. Distances between the points are not indicated — they can simply be measured by applying a ruler to Figure 8.7. It is very difficult to force Mathcad to draw the trajectory of the motion of point 5 (or points 4 — line 4-5 is strictly vertical). There are different ways of realize this closed dotted curve. It is possible, for example, to manually ("worker-peasant approach") tabulate the "closed" function: by changing the angle θ values from 0° to 360° in 10° increments, and recording the obtained coordinates of the point in two vectors with 37 elements, and then to display these two vectors in the plot, connecting them with a dotted line¹. Point 4 moves synchronously with point 5, point 0 describes an arc of a circle, point 3 makes a circular motion, and points 1 and 2 are fixed. In this problem, the main thing is to choose the lengths of the levers so that the trajectory of the points 4 and 5 on a certain section is closest to the horizontal line, and the entire plantigrade machine (and it has two pairs of such legs moving with a shift of 180°) moves without tangible vertical fluctuations.

Tchebyshev had many followers in terms of creating plantigrade machines, translating the rotational motion into a smooth horizontal motion. The most famous of them is the Danish Theo Jansen, who designed and built a machine, whose scheme is shown in Figure 8.8. In this machine, it is possible also to change the lengths between the points (the joints of the "foot") so

¹ The "Intelligent approach" will be to form a user function of the problem solution that constructs a closed dotted curve of the trajectory of the point motion at each step. This approach is implemented in the author's animation of an oscillating cylinder steam engine – see PTC Community forum.

that the motion of the "stop" reference point is as horizontal as possible. In a Jansen plantigrade machine, only points 0 and 1 are fixed, the rest are either in a circular motion or in a more complex movement (see the dotted lines in Figure 8.8). Tchebyshev also had many predecessors in terms of creating plantigrade machines, but P.L. Tchebyshev was the first who, when constructing such mechanisms, began to use a complex mathematical apparatus. Prior to Tchebyshev, such devices were made empirically — adjusting the length of the rods. Figure 8.7 shows a device that "draws" a certain oval, flattened from one side. However mathematicians have proved that it is possible to create a more complex similar mechanical device, which, for example, could reproduce the signature of any person...

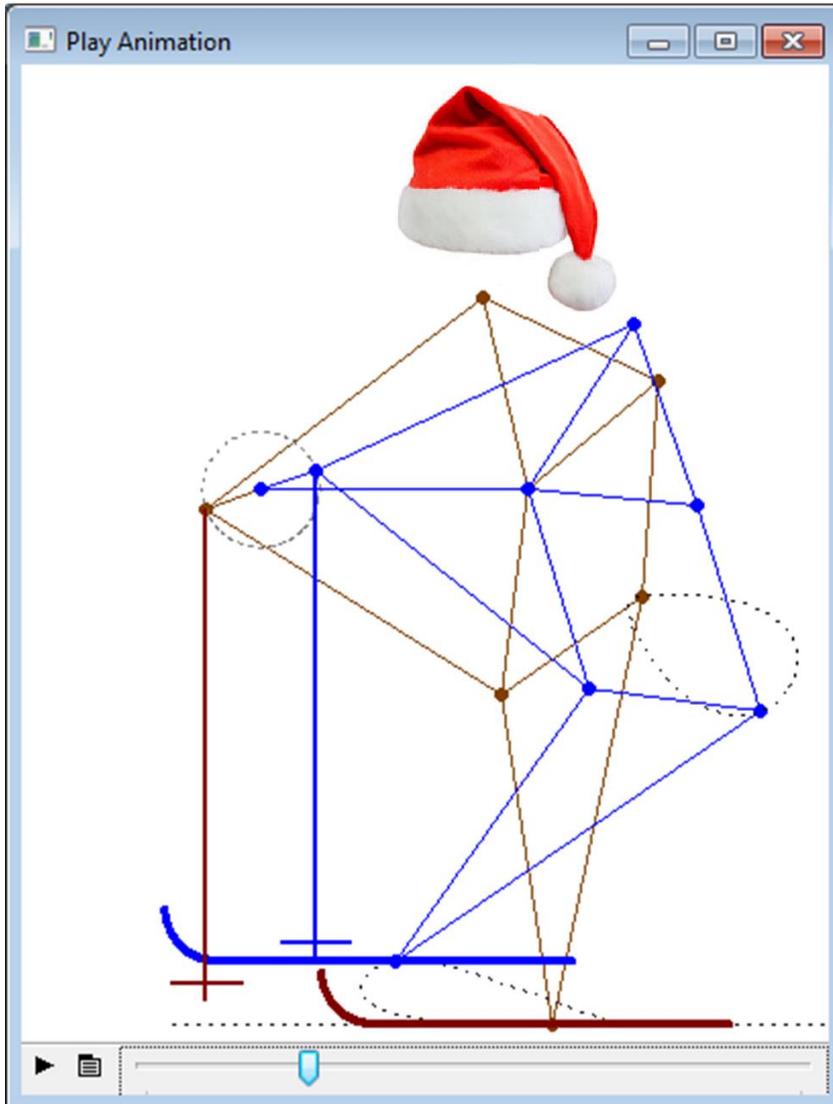


Fig. 8.8. The Jansen plantigrade machine

Figure 8.8 shows the two "legs" of Jansen, one of them moves with a lag of 180° . Theo Jansen built a machine out of wood with a dozen legs, which are driven by a windmill and with a screech move along the beach. This video can be found on YouTube. We have attached a hat, skis and ski poles to this animation. It turned out a good animated Christmas card.

A few words about this site. This site is created and maintained by PTC (www.ptc.com) — the owner of the Mathcad package. Part of this site is given to the community of Mathcad users so that they can communicate with each other, ask questions, keep blogs, share useful information,

etc. On this site, you can create so-called groups for specific topics. One of these "groups" has a name that coincides with the name of this Study. From the same site, it is possible to download a free trial version of the Mathcad package.

The reader himself can use the words "steam machine", "Tchebyshev's plantigrade machine", "Theo Jansen" and others to find animations on the Internet, described in the Study. Though it's one thing to look for and to launch Internet animations, and it's another thing to create them in virtual or real form!

When creating animations in the environment of Mathcad with its rich mathematical apparatus, it is possible to solve not only algebraic equations (this was discussed in Study 4), but also *differential equations and systems* (Study 9). In this case, it is permissible to take into account not only the *kinematics* of models, but also their *dynamics*. The mechanism points of such models will move not only due to the rigid links shown in the Studies, but also under the influence of gravity, inertia and other real physical forces. Thus, in Study 9, it is described how it is possible to simulate and to animate in Mathcad the motion of planets and their satellites under the influence of inertia and of the force of universal gravitation, and the motion of a pendulum under the action of the same inertia force, the force of gravity, the tension of the thread (rigid or elastic), and the resistance force of the viscous medium, where the pendulum swings.

One more useful application of animation is the creation of clips explaining the essence of solved problems — see Figure 10.7 (Animation frames for making boxes from a workpiece) in Study 10.

The Arm of Hella

«She began to hurry, stuck her red-haired head through the vent, reached her arm down as far as she could, her nails clawing at the

lower latch and shaking the frame. Her arm began to lengthen, rubber-like, and became covered with a putrid green. Finally the dead woman's green fingers got hold of the latch knob, turned it, and the frame began to open»¹

M.A. Bulgakov "Master and Margarita"

Figure 8.9 shows an animation frame of the so-called rule of Tchebyshev — a device with rigid connections (links) that converts the circular motion of the point 1 into the almost rectilinear oscillatory motion of the point 4. When implementing this model in the Mathcad 15 environment, the built-in function `Find` is used to solve systems of nonlinear algebraic equations describing the relationships in the mechanism. The lengths of the rods must be selected or calculated so that the maximum straightness of the motion of point 4 is observed. If at some angle of the lever 0 - 1 the length of the rod 2 - 3 is not enough, then the real mechanism is stalled, while in the mathematical model, the function `Find` returns an error message indicating that there is no solution. However if the function `Find` is replaced by the `MinErr` function, then there us no stalling in the mechanism and no errors in the calculation, but there will be... the extension of the rod 2 - 3 (from 30 to 35.8 cm), like... Hella's harm from Bulgakov's novel (Figure epigraph to the divertissement). The `MinErr` function is described in Study 4 (Figure 4.20).

¹ From Chapter 14, Glory to the Cock, PENGUIN BOOKS, 1997 Master and Margarita translated by RICHARD PEVEAR AND LARISSA VOLOKHONSKY ENGLISH TRANSLATION AND NOTES.

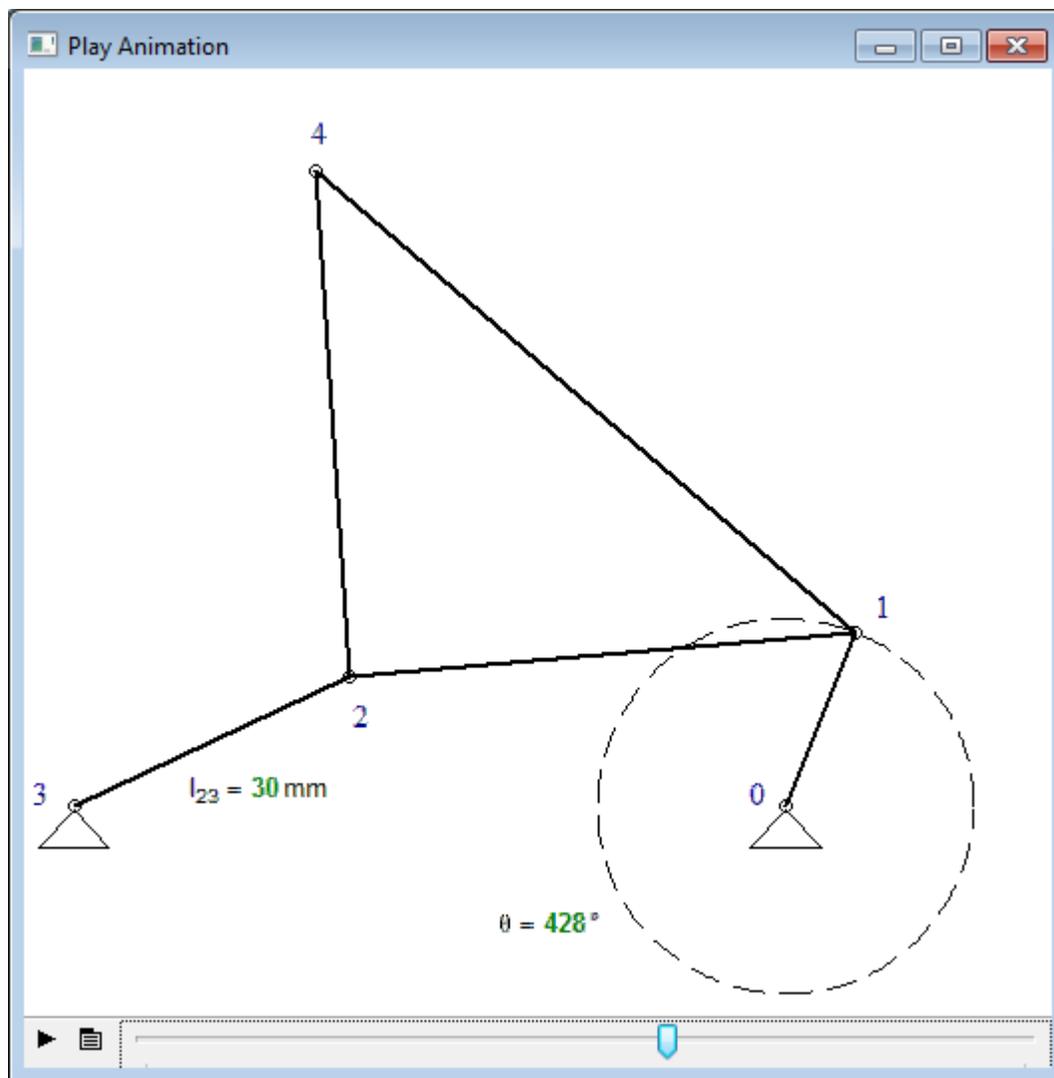


Fig. 8.9. Normal length of the rod 2 - 3 of Tchebyshev rule

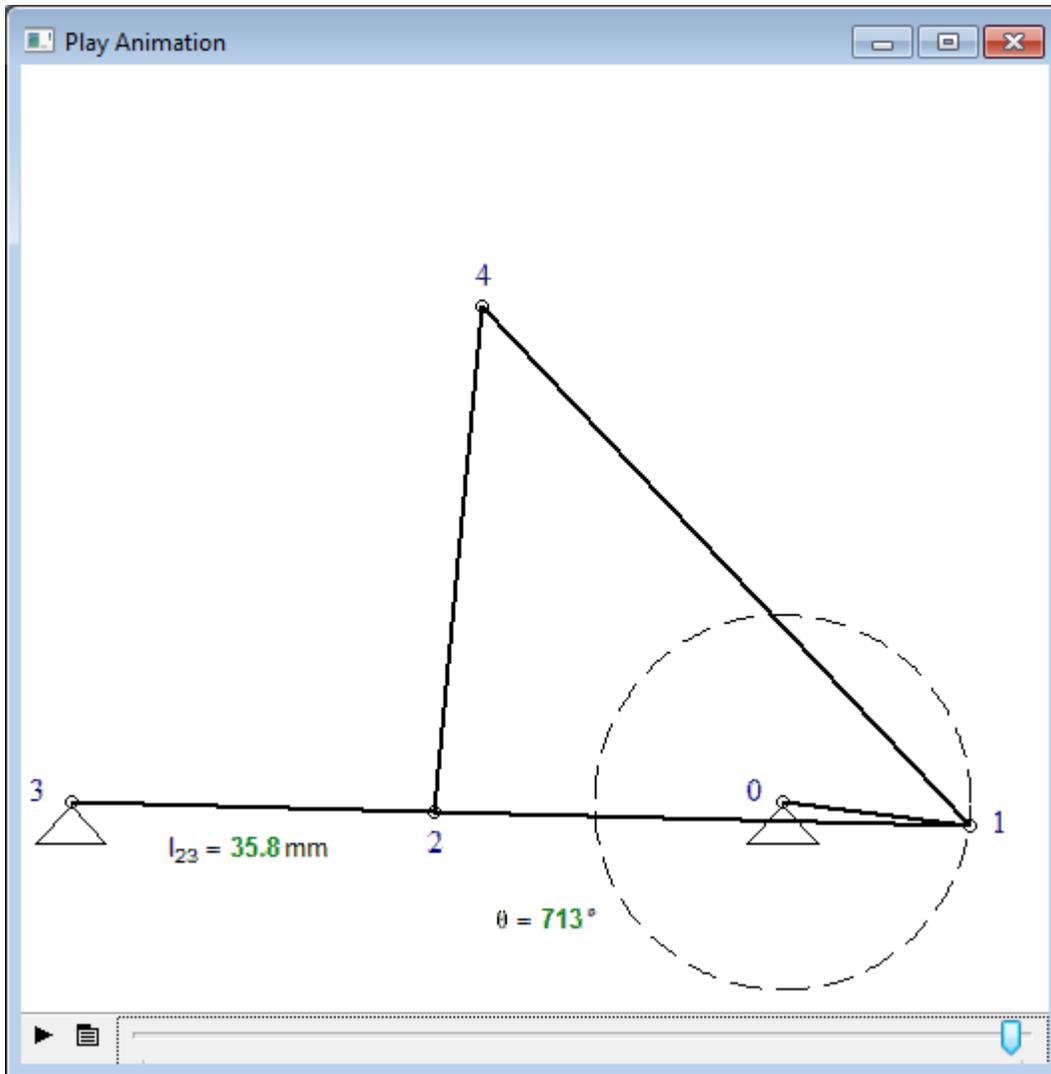


Fig. 8.10. Extension of the rod 2 - 3 of Tchebyshev rule

Elongation or shortening of the rod in a kinematic scheme (using a "soft" rather than rigid connection) can substantially smooth out the course of the same plantigrade mechanisms, that do not always have to walk along a purely rectilinear surface. This approach, by the way, has been introduced long ago in "self-wheeled carriages" — cars and motorcycles, equipped with leaf springs, coil springs and shock absorbers.

Divertissement 2. Journey into the world of science and art on the plantigrade machine

The site of the divertissement <https://community.ptc.com/t5/PTC-Mathcad->

Questions/Tchebyshev-Math-Journey/td-p/500239.

"I sat down and went. In the railway carriage it is quiet, there are no pushes. "

F. M. Dostoevsky "The Gambler"

A century and a half ago only very rich and physically healthy people could afford to travel the world. But with the advent of modern vehicles, such pleasure became available to the "broad masses of working people", and not just to the elected: to go on a plane, car or train — and in a short time comfortably to reach every corner of the Earth¹.

Something similar can be said about... mathematics. Previously, only selected people could get into its jungle — people with special mathematical abilities (with a special "mathematical ear") and having the appropriate mathematical education. But at the present time the circle of such elect has significantly expanded due to the emergence of... computer mathematical packages that facilitate the journey into the world of mathematics. Conditionally it can be said that some kind of mass mathematical tourism has appeared.

And one more important introductory remark.

Why do we study mathematics at schools and universities?

First, in order to be able to master other academic disciplines: physics, chemistry, theoretical mechanics, fluid dynamics, material resistance, engineering graphics, economics, finance, etc. That's why the course of mathematics is read at the very beginning of study at universities!

Secondly, one must always remember that mathematics is the best gymnastics (fitness)

¹ The words in the epigraph are pronounced by the paralyzed "granny" who "lost her legs" and who is transferred from place to place in a chair. She quietly lived in Moscow (the second half of the XIX century), but dispersed the doctors, boarded the train and "fell down like a snow on her head" to her relative — the general who was in Roulettenberg (the German city of Baden-Baden), waiting for the death of the "granny" in order to get her inheritance – see <http://ilibrary.ru/text/67/p.1/index.html>.

for the mind. Studying mathematics, we develop our mental abilities, which will be useful for us in solving not only purely mathematical, but also everyday problems.

And thirdly, the study of mathematics (a journey into its "jungle") is in itself an interesting and fascinating activity that can be practiced as a hobby. But without mathematical computer packages, it was almost impossible for ordinary people to do this, if, we repeat, there are no special mathematical talents and the corresponding mathematical education.

Let's illustrate these theses by simple and entertaining mechanical-mathematical examples.

The Polytechnic Museum of Moscow keeps a device that has a direct relation to travel and "around the world" and "in mathematics." This is the Tchebyshev plantigrade machine (the plantigrade machine), which converts rotational motion into translational motion — see Figure 8.11.

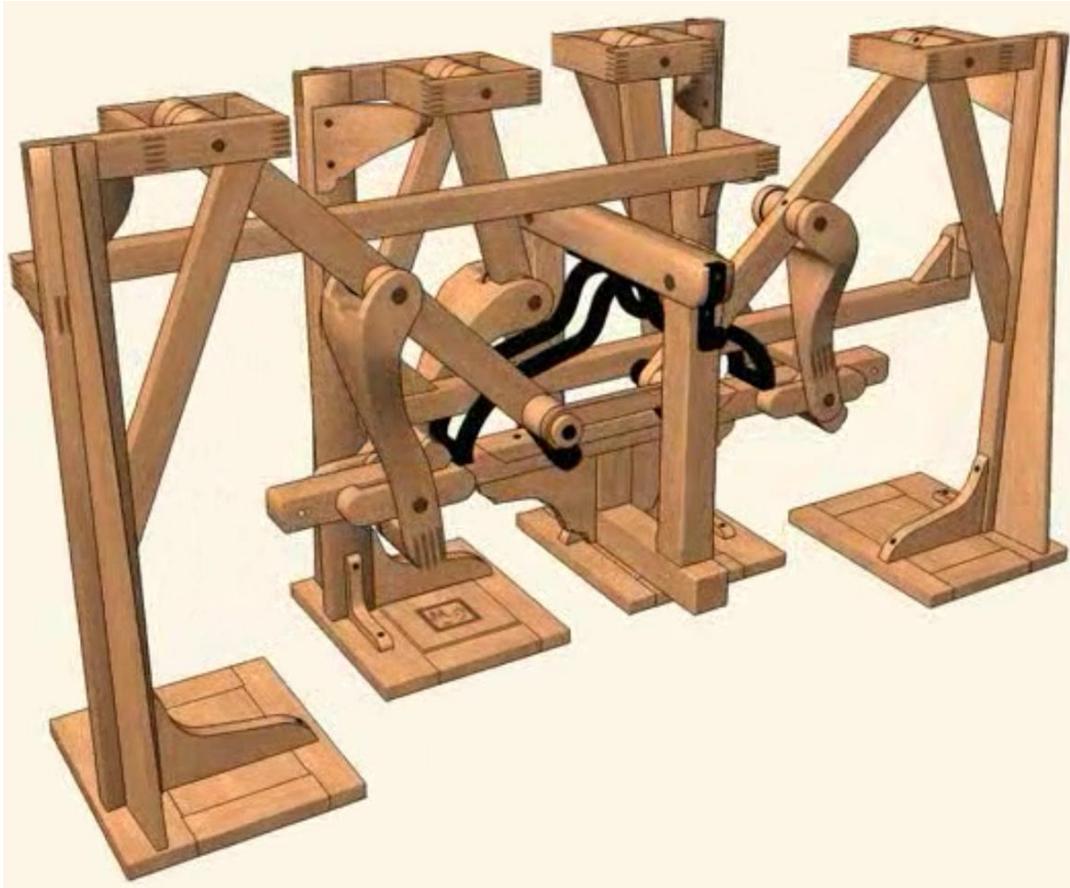


Fig. 8.11. Tchebyshev's plantigrade machine

Figure 8.11 is taken from the site of mathematical studies <http://www.Studies.ru/ru/Studies/Tchebyshev-plantigrade-machine>, where it is possible to see the animation of the motion of this machine, as well as a detailed description of its device. The central link of this exotic vehicle is the mechanism schematically shown in Figure 8.12 (it is also called λ -mechanism). Points 1 and 2 are fixed, point 3 rotates around point 2, point 0 moves along an arc of a circle, and point 4 on a closed curve ("mushroom's head"), the lower part of which is close to a straight horizontal line. The movement of this point is transmitted to the "feet" of the machine shown in Figure 8.11.

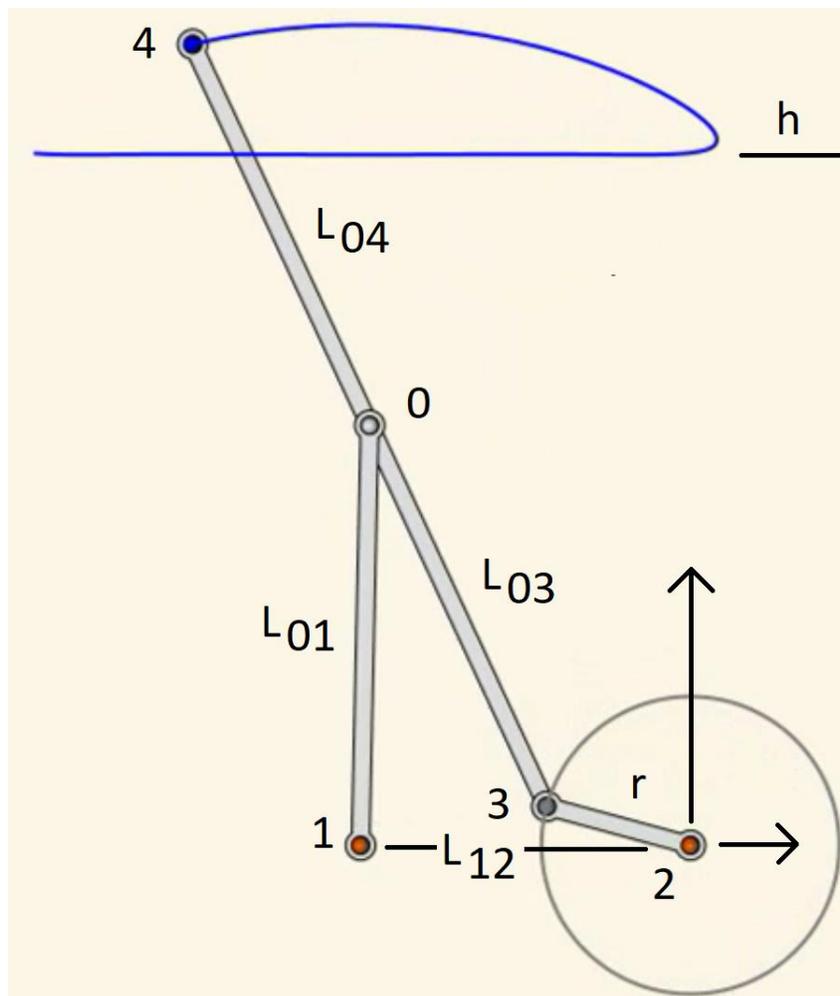


Fig. 8.12. Scheme of the central mechanism of the plantigrade machine

These devices were made even before Tchebyshev, choosing *empirically* the length of the segments L_{01} , L_{03} , L_{12} and L_{04} , as well as the height h so that the plantigrade machine went without vertical pushes. But only this great Russian begun to apply *higher mathematics* for the design of such devices. We can say that Tchebyshev saddled this "wooden horse" and made a trip into the World of Mathematics. Let's do it. For us, we repeat, there is no need of special mathematical talents and corresponding mathematical formation because we have at hand a computer with the mathematical program Mathcad. We will solve the problem in a *hybrid* way (see Study 25) — using both analytical transformations (computer *symbolic mathematics*) and *numerical* (approximate) methods of solution.

A little about numerical mathematic.

At present, numerical mathematics replaces symbolic mathematics, the mathematics of analytical evaluations, in solving not only purely practical problems, but also in educational purposes. "Culprits" of this process are computers. A vivid concrete example. If we want to embody our plantigrade machine in metal or wood (see Figure 8.11), then we will have to calculate not only its geometry (see Figure 8.12 and other figures below), but also the shape and the dimensions (profiles) of its parts. They should be strong enough that the machine does not break when walking, but not too massive, so that the machine easily moves. Calculations of this kind are conducted within the framework of the scientific discipline "Strength of materials". Previously, Russian students of technical universities even had such a saying: "I passed the exam on Strength of materials so I can marry!". The fact is that this academic subject was very difficult to master because it was overloaded with complex empirical formulas, difficult for both understanding and for simple mechanical memorization. There was a special "terrible" formula or set of formulas for each concrete case of structural strength calculation. Our days, the *finite element method* (FEM) makes it possible to split part of any complexity into individual small (finite) elements, for the calculation of these elements are applied fairly simple and understandable formulas. This work became possible only after the appearance of high-speed computers with the corresponding programs. This revolutionary shift from analytical transformations to numerical computer calculations has affected other scientific disciplines — hydrodynamics, heat and mass transfer and etc. However there are still "partisans" of the "rules of thumbs" and "back of envelope calculations" for the engineering practitioners.

The core of the calculation of the plantigrade machine is the following: it is necessary to calculate the lengths of the segments L_{01} , L_{03} , L_{12} and L_{04} , as well as the height h for a given

value of the radius r so that point 4 on the lower part of its trajectory moves almost along a straight horizontal line, consequently a plantigrade machine with such a central mechanism would go as far as possible without vertical pushes.

Figure 18.13 shows the beginning of the Mathcad-sheet with the calculation of the geometry of the mechanism shown in Figure 8.19. In the first line of the sheet we see the input of the initial data — guess values of the lengths of the links of the central mechanism of the plantigrade machine¹. The value of the angle α of turning the link 2 - 3 is tied to the system variable `FRAME`, which will change from 0 to 360 and it will determine the frames of the animation, which we will create later. When debugging the calculation associated to a graph (a separate animation frame) with the image of our mechanism (see below), we will assign (global assignment) a value of 30° to `FRAME` and see the geometry of the mechanisms when turning to 30 degrees the lever 2 - 3. When creating the animation, it is necessary to disable this operator so that the Mathcad package can create the animation. The package itself will change the value of the variable `FRAME` and store the animation frames in the avi-container².

Points 1 and 2, we repeat, are fixed, while points 0, 3 and 4 are in motion. Their coordinates are given in the form of functions with names x_0 , y_0 , x_3 , y_3 , x_4 and y_4 with arguments α , r , L_{12} , L_{03} , L_{01} and L_{04} . Working with functions, rather than with the correspondent scalar values, allows us, firstly, to create graphs and animations with the drawing of the trajectories of

¹ They are entered with meter as unit of length. However this is a convention. There may be other meanings, and other units of length (centimeters, fots, inches or arshins with vershoks, which were in use in Russia at the time of Tchebyshev: 3 arshins = 7 fots, 16 vershok = 1 arshin). The main thing is that this method allows you to control the dimensions, enter and output lengths with the necessary units of length.

² Audio Video Interleave : A container or wrapper format is a metafile format whose specification describes how different elements of data and metadata coexist in a computer file.

all points¹ and, secondly, to carry out *optimization* — to find the optimal lengths L_{12} , L_{03} , L_{01} and L_{04} , as well as the height h , which ensure the maximum rectilinear motion of the plantigrade machine — movement without "pushes".

$$r := 15\text{m} \quad L_{12} := 35\text{m} \quad L_{03} := 35\text{m} \quad L_{01} := 35\text{m} \quad L_{04} := 35\text{m} \quad h := 55\text{m} \quad \alpha := \text{FRAME deg}$$

$$x_1(L_{12}) := -L_{12} \quad y_1 := 0\text{m} \quad x_2 := 0\text{m} \quad y_2 := 0\text{m} \quad x_3(\alpha, r) := r \cdot \cos(\alpha) \quad y_3(\alpha, r) := r \cdot \sin(\alpha)$$

$$L_{03}^2 = (x_0(\alpha, r, L_{12}, L_{03}, L_{01}) - x_3(\alpha, r))^2 + (y_0(\alpha, r, L_{12}, L_{03}, L_{01}) - y_3(\alpha, r))^2$$

$$L_{01}^2 = (x_0(\alpha, r, L_{12}, L_{03}, L_{01}) - x_1(L_{12}))^2 + y_0(\alpha, r, L_{12}, L_{03}, L_{01})^2$$

$$x_0(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{x_3(\alpha, r)^2 + L_{01}^2 - L_{03}^2 + y_3(\alpha, r)^2 - x_1(L_{12})^2 - \frac{y_3(\alpha, r) \cdot [y_3(\alpha, r)^3 - x_1(L_{12}) \cdot \sqrt{L_{01}^2 + 2 \cdot L_{01} \cdot L_{03} + L_{03}^2 - x_1(L_{12})^2} + 2 \cdot x_1(L_{12}) \cdot x_3(\alpha, r)]}{2 \cdot y_3(\alpha, r)}}{2 \cdot y_3(\alpha, r)}$$

$$y_0(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{y_3(\alpha, r)^3 - x_1(L_{12}) \cdot \sqrt{L_{01}^2 + 2 \cdot L_{01} \cdot L_{03} + L_{03}^2 - x_1(L_{12})^2} + 2 \cdot x_1(L_{12}) \cdot x_3(\alpha, r)}{2 \cdot y_3(\alpha, r)}$$

$$a(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{x_3(\alpha, r) \cdot y_0(\alpha, r, L_{12}, L_{03}, L_{01}) - y_3(\alpha, r) \cdot x_0(\alpha, r, L_{12}, L_{03}, L_{01})}{x_3(\alpha, r) - x_0(\alpha, r, L_{12}, L_{03}, L_{01})}$$

$$b(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{y_3(\alpha, r) - y_0(\alpha, r, L_{12}, L_{03}, L_{01})}{x_3(\alpha, r) - x_0(\alpha, r, L_{12}, L_{03}, L_{01})}$$

$$x_4(\alpha, r, L_{12}, L_{03}, L_{01}, L_{04}) := x_0(\alpha, r, L_{12}, L_{03}, L_{01}) + \frac{L_{04}}{L_{03}} \cdot (x_0(\alpha, r, L_{12}, L_{03}, L_{01}) - x_3(\alpha, r))$$

$$y_4(\alpha, r, L_{12}, L_{03}, L_{01}, L_{04}) := a(\alpha, r, L_{12}, L_{03}, L_{01}) + b(\alpha, r, L_{12}, L_{03}, L_{01}) \cdot x_4(\alpha, r, L_{12}, L_{03}, L_{01}, L_{04})$$

Fig. 18.13. Beginning of the calculation of a plantigrade machine

The position of point O (the "central" point of our mechanism) is determined through the analytical (symbolic) solution of a system of two algebraic nonlinear equations written at the beginning of Figure 8.13. These equations were solved symbolically in a separate Mathcad-sheet (it is also available on the Study website). The system of equations has two roots, one of them was transferred to the sheet in Figure 8.13 and formed the functions named x_0 and y_0 . The

¹ It is possible to find many drawings and animations of various kinematic devices on the Internet. However all of them, as a rule, are given without drawing the trajectories of the motion of the linkage points (joints).

second extra root captures the geometry of the mechanism shown in Figure 8.12, if point 0 is not higher, but lower than points 1 and 3. These roots (algebraic expressions) are very voluminous: Figure 8.13 shows only their beginning¹.

The point 4 is on a straight line passing through points 0 and 3. To determine the coordinates of the point 4, use the equation of the straight line $y = a + b \cdot x$, whose coefficients a and b are also symbolically determined in a separate document through the solution of a system of two algebraic equations and carried over in the document in Figure 18.13. This Mathcad-sheet is also posted on the Study's website.

After the coordinates of all five points of our kinematic scheme have been specified or calculated, it is easy to display it on the graph — see Figure 8.14.

¹ In the calculation, it could be simpler to use not symbolic mathematics and not user functions, but numerical mathematics (the Given-Find block) and scalar user values. However in this case, we repeat, it would be impossible to draw the trajectories of the motion of the points, to create an animation with the display of the trajectories of the points movement and their optimization.

are being solved more and more often not analytically, but numerically using a computer.

On the lower part of the "mushroom cap" (Figure 8.14), points are marked that indicates the rotation of the link 2 - 3 along -100, -90, -80,... 0, 10, 20,..., 90 and 100 angular degrees. These angles are stored in a program-created vector named A. It is necessary to make sure that these points are not in an arc, but in a straight line. In order to obtain this, it is simple, for example, to minimize the sum of the squares of the deviations of these points from the horizontal line!

$$S(L_{12}, L_{03}, L_{01}, L_{04}, h) := \sum_{i=0}^{20} (y_4(A_i, r, L_{12}, L_{03}, L_{01}, L_{04}) - h)^2$$

$$S(L_{12}, L_{03}, L_{01}, L_{04}, h) = 341.443 \text{ m}^2$$

$$\begin{pmatrix} L_{12} \\ L_{03} \\ L_{01} \\ L_{04} \\ h \end{pmatrix} := \text{Minimize}(S, L_{12}, L_{03}, L_{01}, L_{04}, h) = \begin{pmatrix} 31.233 \\ 39.39 \\ 39.384 \\ 39.362 \\ 63.804 \end{pmatrix} \text{ m}$$

$$S(L_{12}, L_{03}, L_{01}, L_{04}, h) = 0.035 \text{ m}^2$$

Fig. 8.15. Optimization of the plantigrade machine

Figure 8.15 shows how the objective function with the name S and with five arguments — *optimization parameters* is specified. The function S returns the mentioned sum of squares of deviations of points from the horizontal line. With the optimization parameters specified by the first line in Figure 8.13, this amount is equal to 341.5 m². It is possible to request to the Mathcad package to change the optimization parameters L₁₂, L₀₃, L₀₁ and L₀₄ and h so that this amount becomes minimal. This "request" is performed by the Mathcad built in function

Minimize: the sum of the squares of deviations of points from the line became 0.035 m^2 , and our plantigrade machine began to go almost without pushes — see Figure 8.16.

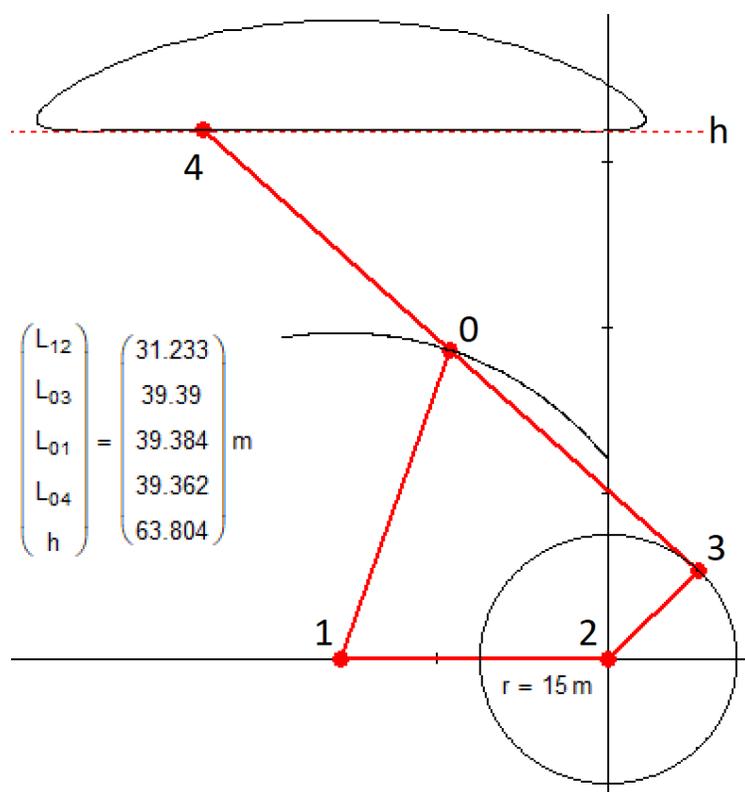


Fig. 8.16. Optimized central mechanism of the Tchebyshev's plantigrade machine.

Figure 8.16 shows that after optimization the values of variables L_{03} , L_{01} and L_{04} became almost equal to each other. The conclusion is that they are in fact equal, and that Tchebyshev has proved this analytically long ago. So the number of independent parameters of our optimization problem can be reduced from five to three. This reduction is important for manual analytical calculations. Today, when we increasingly solve similar problems on the computer, this *optimized optimization* becomes less important.

Figure 8.17 shows four frames of the animation for different values of the angle of rotation α . The same animation, we repeat, could be seen on the site

became abroad the most famous Russian mathematician.

<https://community.ptc.com/t5/PTC-Mathcad-Questions/Tchebyshev-Math-Journey/td-p/500239>.

There the reader can find the corresponding Mathcad-sheets in two versions — Mathcad 15 and Mathcad Prime.

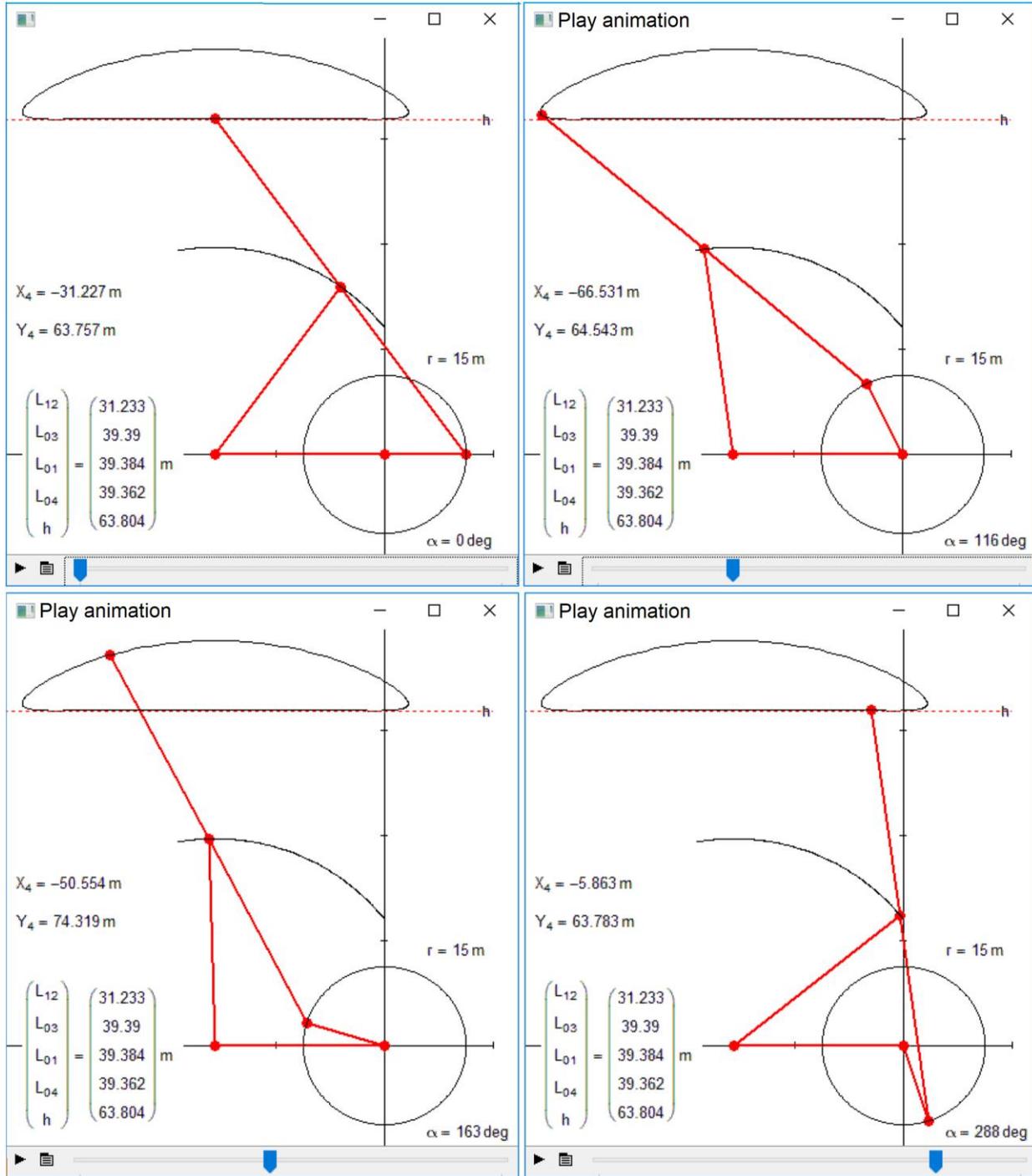


Fig. 8.17. Frames of animation of the central mechanism of the Tchebyshev's plantigrade machine

Let's move ahead.

On the pantograph machine a person can travel not only to the world of science (mathematics and mechanics — see above), but also to the world of... art, without having the appropriate talents. And with the advent of computers, in particular computer graphics, new forms of visual art emerged, accessible to the same "broad masses of working people". We will propose one more form of fine art connected with our linkages!

Figure 8.18 shows how the two links pivoted at one end at hinges 1 and 2 (fulcra of the levers in a more dynamics language), and rotating around these points, move three other links (one of them slides in support 3) and... draw a funny face. It is possible to change the parameters of this linkage and... make the face look more cute or more amusing...

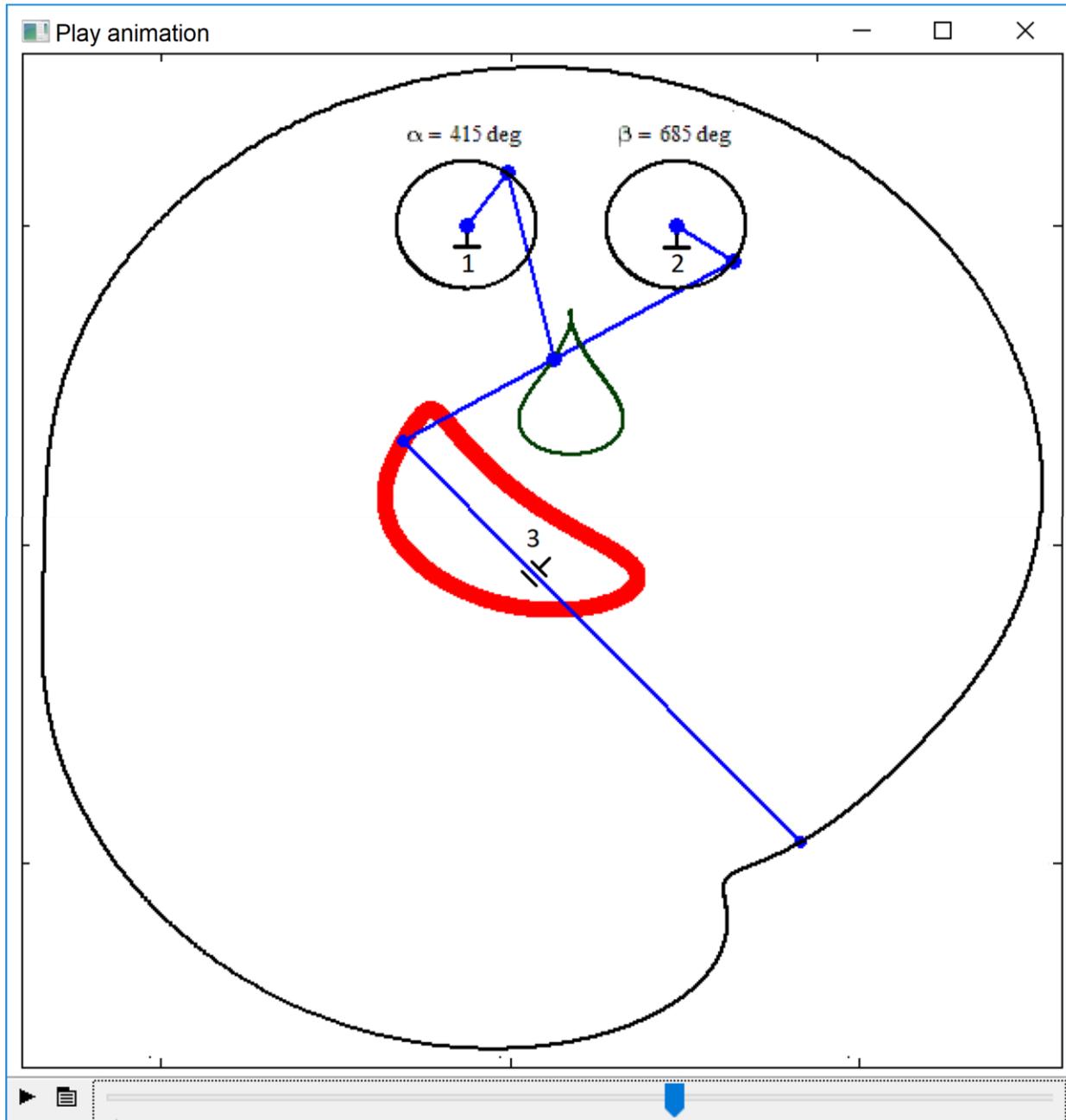


Fig. 8.18. Drawing a funny face with a five-link kinematic device

If an "eye" in Figure 8.25, the lever rotating around a fixed point is replaced with a more complex device — with a cross-shaped joint, then it is possible to draw... a bird (The Bluebird of Happiness) — see Figure 8.19

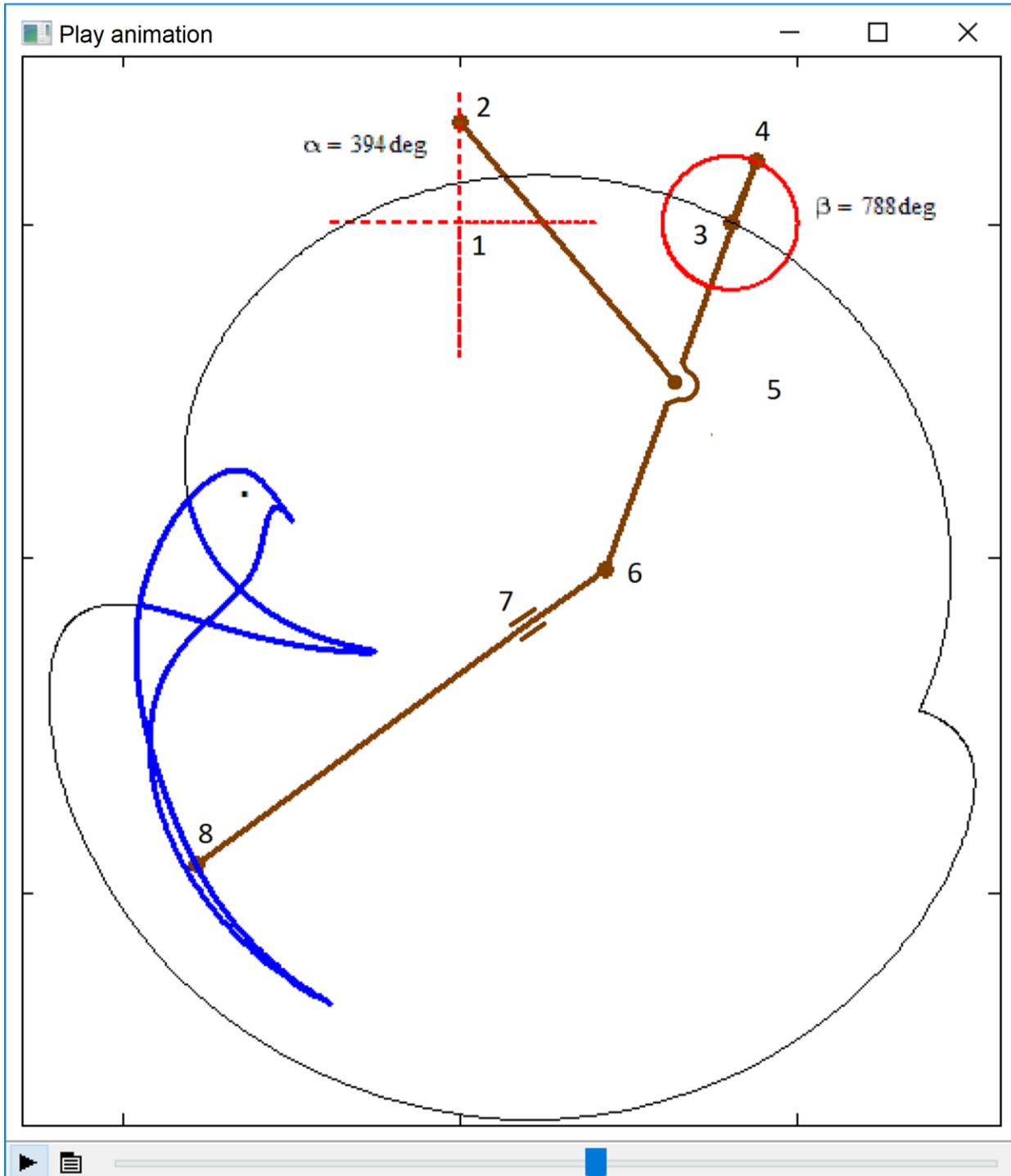


Fig. 8.19. Drawing a bird with the help of a kinematic scheme

Points 1, 3 and 7 are fixed. Point 2 moves right-left, up-down, left-right and up-down, drawing a cross. Point 4 moves in a circular orbit. Points 5 and 6 write out intricate trajectories

that can be seen on the site's website¹. The final point 8 draws our blue bird.

Figure 8.20 shows a more complex hinge mechanism that depicts a certain lace: with four links that rotate around four hinges, other articulated links are attached to them, the joints of which draw closed curves. To this design, it is possible to add new hinges and to weave a "lace" — see the site <https://community.ptc.com/t5/PTC-Mathcad-Questions/New-art-form-with-Mathcad/mp/537923> with the corresponding drawings, animations and Mathcad-files.

¹ <https://community.ptc.com/t5/PTC-Mathcad-Questions/Square-circle-lemon/m-p/538487>

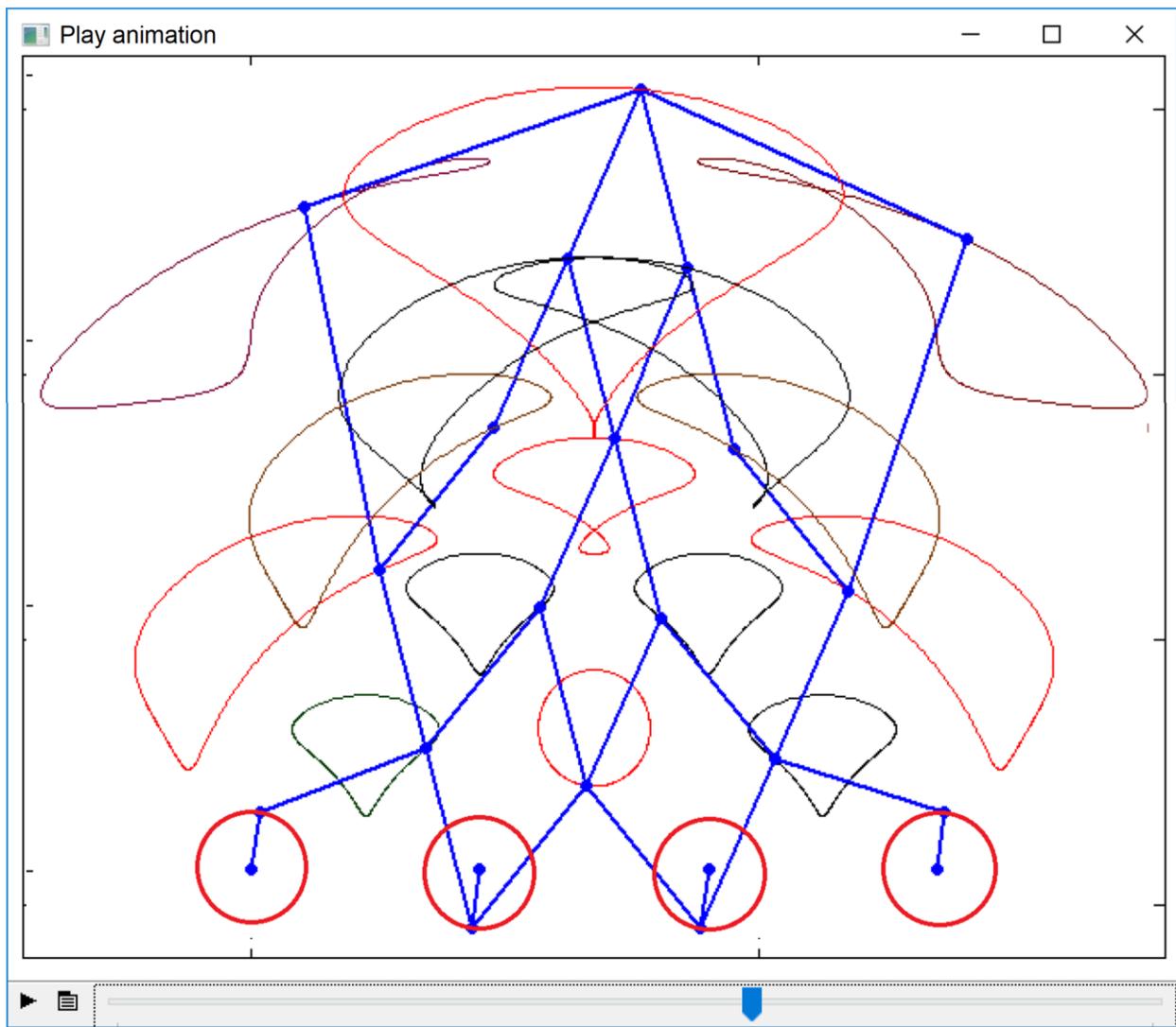


Fig. 8.20. Lace of the trajectories of the joints of a linkage

The number of levers that rotate around points can be increased, the radii of rotation, the speed of rotation, the lengths of the connected links, and so on can be changed. However it is possible to limit oneself to only two initial levers, and to rotate them not only in a circle, but also in... a "square" — see Figure 8.28.

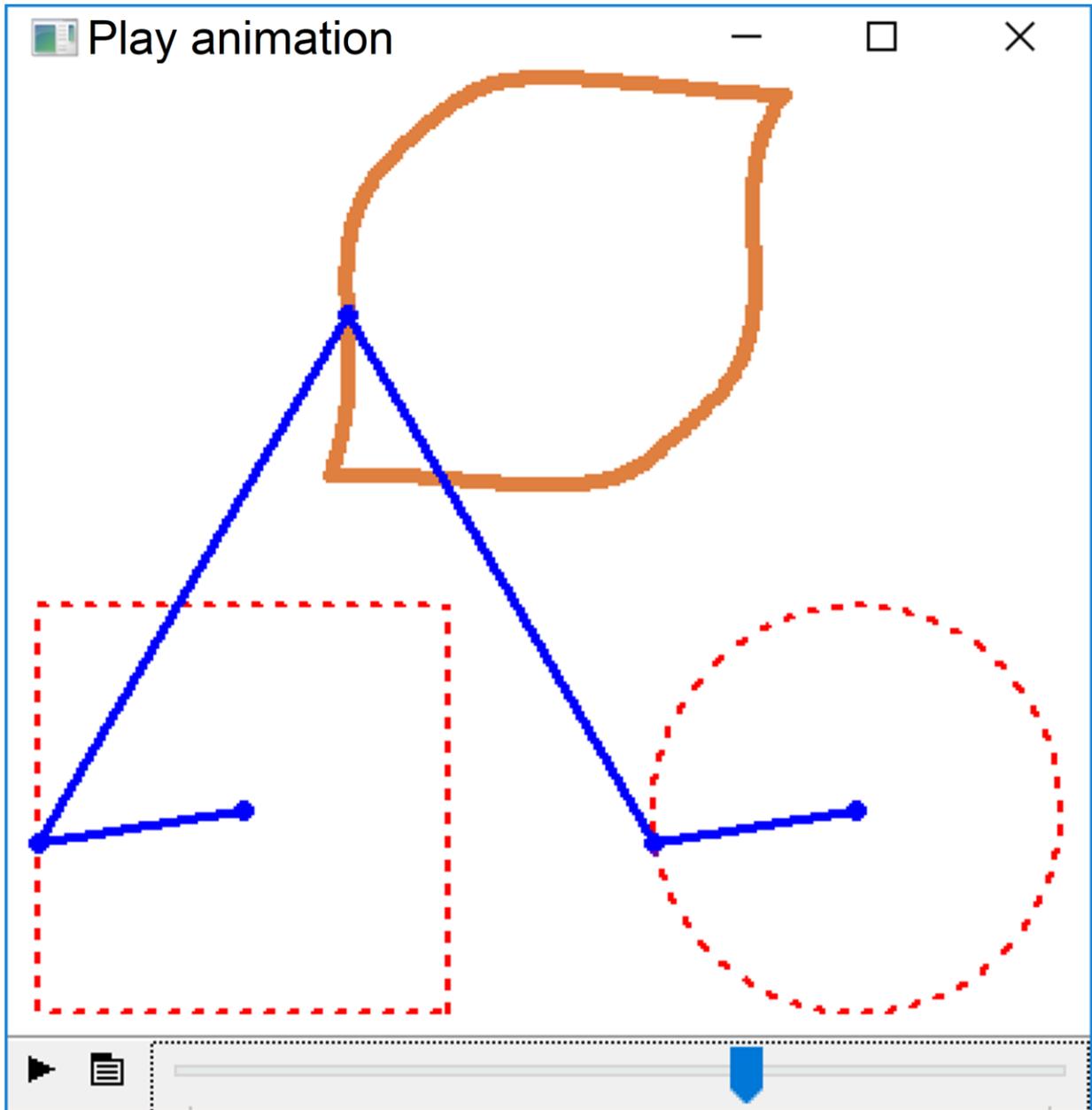


Fig. 8.21. Drawing the contour of a lemon with the help of a linkage

The levers shown in Figure 8.21 draw a contour of... a lemon. Instead of a square, it is possible to use a triangle or other polygons and build entertaining and in some way beautiful closed curves. These attempts by the author and his colleagues are shown on the forum with the name "Circle + square = lemon" (<https://community.ptc.com/t5/PTC-Mathcad-Questions/Square-circle-lemon/m-p/538487>).

And finally.

In Mathcad Prime, alas, there are no tools for creating animations. However this problem can be solved in such a way — by creating several picture frames, which then can be used to create animation with tools that are available on the Internet. With this approach, for example, the author's animation of the chain sagging (Study 7 theme) was created, consisting of two sections with different specific (linear) mass and pulled up at one end — see the PTC Community forum.

Assignments for the readers:

1. Create an animation of the steam engine shown in Figure 8.1, and of the steam engine mentioned in footnote 10.
2. Change the length of the levers of Tchebyshev rule, shown in Figure 8.9, so that point 4 moves as straight as possible. One of the solutions is to put several points on the assumed straight line and reduce the solution to the search of the root of a system of equations. We will use this approach for the determining the coefficients of the second-order curve equation describing the motion of a cosmic body near the Earth (see Figures 9.16-9.18 in Study 9).